Newton's Method (Newton-Raphson)

In this method we suggest the formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i=0,1,2,\dots$$

It is a very fast root-finding method and useful when $f'(x)$ is not difficult to evaluate

Inorder to solve any example we will write the following steps;

1- Start with an initial guess x_0

2- Calculate the value of $[f'(x)]$. (requires the knowledge of the derivative f' of the function f)

3- Calculate

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i=0,1,2,\dots$$

until $\left|\frac{f(x_i)}{f'(x_i)}\right| < \varepsilon$ or

$$|x_{i+1} - x_i| < \varepsilon$$

Ex Solve for the value of X

$f(x) = x^3 - x - 1$ with $x_0 = 1$

Solution

$f'(x) = 3x^2 - 1$

we have ; $x_0 = 1$, $f(x_0) = -1$, $f'(x_0) = 2$

$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = 1 - \frac{-1}{2} = 1.5 = x_1$

$x_1 = 1.5$, $f(x_1) = 0.875$, $f'(x_1) = 5.75$

$x_2 = 1.3478$, $f(x_2) = 0.1005$, $f'(x_2) = 4.4496$

$x_3 = 1.3252$, $f(x_3) = 0.002$, $f'(x_3) = 4.2684$

$x_4 = 1.3247$, $f(x_4) = \bar{0}.00007 \simeq 0$, $f'(x_4) = 4.2644$

$\therefore x_5 = 1.3247 - \frac{0}{4.2644} = 1.3247$

$\therefore$ we will stop $\left|\frac{0}{4.2644}\right| < \varepsilon$

and

$x_4 = x_5 = 1.3247$

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Ex Solve $f(x) = x^2 - 2$ $\quad x_0 = 1$

Solution $f'(x) = 2x$

$x_0 = 1$, $f(x_0) = -1$, $f'(x_0) = 2$

$x_1 = 1.5$, $f(x_1) = 0.25$, $f'(x_1) = 3$

$x_2 = 1.4166$, $f(x_2) = 0.0069$ $\quad f'(x_2) = 2.8333$

$x_3 = 1.4142$,

$\vdots$

Continue until $|x_k - x_{k-1}| < \varepsilon$

Ex Find the smallest positive root for the equation $x \sin x = 1$ with $x_0 = 1$

Solution we write

$$x \sin x - 1 = 0 \Rightarrow \sin x - \frac{1}{x} = 0$$

$\therefore f(x) = \sin x - \frac{1}{x}$

$f'(x) = \cos x + \frac{1}{x^2}$

$x_0 = 1$, $f(x_0) = -0.1585$, $f'(x_0) = 1.5403$

$$x_1 = 1 - \frac{-0.1585}{1.5403}$$

$\Rightarrow$

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$x_1 = 1.1029$ $\quad f(x_1) = -.0141$, $f'(x_1) = 1.2731$

$f(x_2) = -.0003$ $\quad f'(x_2) = 1.2471$

$x_2 = 1.1139$

$x_3 = 1.1141$ $\quad f(x_4) = -.00007$

$\simeq 0$

stop

$x_5 = x_4 = 1.1141$

<u>Example</u>: Use the Newton-Raphson method to estimate the root of the function $f(x) = x^2 - 10$, with intial point $x = 3$.

<u>Solution</u>: $f(x) = x^2 - 10$

$$f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{Let } n = 0$$

$$x_1 = 3 - \frac{(3)^2 - 10}{2(3)} \simeq 3.16666$$

$$x_2 = 3.16666 - \frac{(3.16666)^2 - 10}{2(3.16666)}$$

$$\simeq 3.162280702$$

$$x_3 = 3.162280702 - \frac{(3.162280702)^2 - 10}{2(3.162280702)}$$

$$= 3.16227766$$

<u>Example</u>: Let $f(y) = y^3 - 2y - 5$. Use Newton-Raphson method with intial point $y_0 = 2$

Note: compute y_1, y_2, y_3.

<u>Solution</u>:

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}$$

$$= y_n - \frac{y_n^3 - 2y_n - 5}{3y_n^2 - 2}$$

$$= \frac{2y_n^3 + 5}{3y_n^2 - 2}$$

Let $n = 0$

$$y_1 = \frac{2(y_0)^3 + 5}{3(y_0)^2 - 2} = \frac{2(2)^3 + 5}{3(2)^2 - 2} = \frac{21}{10}$$

$$= 2.1$$

Let $n = 1$

$$y_2 = \frac{2(2.1)^3 + 5}{3(2.1)^2 - 2} = 2.094568121$$

$$n = 2 \Rightarrow y_3 = 2.094551482$$

Example: Let $f(x) = e^{2x} - x - 6$. Use Newton-Raphson method to estimate root?

Solution: we need intial guess to start our computation.

Let $x_0 = 1 \Rightarrow f(1) = e^2 - 1 - 6$
$= 0.389$

Let $x_0 = 0.95 \Rightarrow f(0.95) = -0.2641$

and since $f(0.95)$ is closer to 0 than $f(1)$ then may be the root is closer to 0.95.

so Let's take $x_0 = 0.97$

$f(x) = e^{2x} - x - 6$

$f'(x) = 2e^{2x} - 1$

Let $n = 0 \Rightarrow$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.97 - \frac{e^{2\times 0.97} - 0.97 - 6}{2e^{2\times 0.97} - 1}$$

$$= 0.97 + \frac{0.0112490}{12.917501991}$$

$$= 0.970870834$$

let $n = 1 \Rightarrow$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.97087002$$

Here we might say that the root is 0.9709

Numerical Methods To solve Non-Linear Equations

To find the solution of equation like $f(x) = 0$, we can use one of the following

1- Bisection method
2- Newton's method

How to implement Bisection method:

Suppose the function $f(x)$ is continous on the interval $[x_0, x_1]$ and $f(x_0), f(x_1)$ have opposite signs, the different signs of $f(x)$ in the points x_0, x_1 means there is at least one root for our equation in the interval $[x_0, x_1]$. So we bisect the interval to get

$$x_2 = \frac{x_0 + x_1}{2}$$, and either use $[x_0, x_2]$

or $[x_2, x_1]$

If $f(x_0) \cdot f(x_2) < 0 \Rightarrow$ our new interval is $[x_0, x_2]$

<u>else</u> the new interval is $[x_2, x_1]$

$f(x_2) \cdot f(x_1) < 0$

Then we continue subdividing until

$|f(x_i) - f(x_{i-1})| < \epsilon$ or

$|x_i - x_{i-1}| \leq \epsilon$

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<u>Example</u>: Bisect the interval $[1, 2]$ six times to find the root of the equation

$f(x) = x^3 - x - 1$

<u>Solution</u>:

$x_1 = 1 \Rightarrow f(x_1) = -1$

$x_2 = 2 \Rightarrow f(x_2) = 5$

$x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{1+2}{2} = 1.5$

Now, we will construct the following table:

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	1	2	1.5	-1	5	0.875	$x_2 = x_3$
2	1	1.5	1.25	-1	0.875	$\bar{0}.2968$	$x_1 = x_3$
3	1.25	1.5	1.375	$\bar{0}.2968$	0.875	0.2246	$x_2 = x_3$
4	1.25	1.375	1.3125	$\bar{0}.2968$	0.2246	$\bar{0}.0515$	$x_1 = x_3$
5	1.3125	1.375	1.3437	$\bar{0}.0515$	0.2246	0.0823	$x_2 = x_3$
6	1.3125	1.3437	1.3281	$\bar{0}.0515$	0.0823	0.0144	$x_2 = x_3$

the root is $\dfrac{1.3125 + 1.3281}{2} = 1.3203$

and $f(1.3203) = \bar{0}.0187$

Ex Bisect the interval [0.6, 0.8] five times to solve the eq. $2^x - 5x + 2 = 0$

Solution

$x_1 = 0.6 \qquad f(x_1) = 0.5157$

$x_2 = 0.8 \qquad f(x_2) = \bar{0}.2588$

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0.6 + 0.8}{2} = 0.7$$

i	x_1	x_2	x_3	$f(x_1)$	$f(x_2)$	$f(x_3)$	
1	0.6	0.8	0.7	0.5157	$\bar{0}.2588$	0.1245	$x_1 = x_3$
2	0.7	0.8	0.75	0.1245	$\bar{0}.2588$	$\bar{0}.0682$	$x_2 = x_3$
3	0.7	0.75	0.725	0.1245	$\bar{0}.0682$	0.0279	$x_1 = x_3$
4	0.725	0.75	0.7375	0.0279	$\bar{0}.0682$	$\bar{0}.0202$	$x_2 = x_3$
5	0.725	0.7375	0.7312	0.0279	$\bar{0}.0202$	0.004	$x_1 = x_3$

So the root is $\dfrac{0.7375 + 0.7312}{2} = 0.7343$

and $f(0.7343) = \bar{0}.0079$

H.W Use bisection method to solve $x - \sin x - 1 = 0$ [1, 2.4]

Bisection Method :

Example : Let $f(x) = x^2 - 3$, interval $[1,2]$

Solution

a	b	f(a)	f(b)	c	f(c)	up date
1	2	-2	1	1.5	-0.75	a=c
1.5	2	-0.75	1	1.75	0.062	b=c
1.5	1.75	-0.75	0.0625	1.625	-0.359	a=c
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	a=c
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	a=c
1.7188	1.75	-0.0457	0.0625	1.7344	-0.0081	b=c

So our new interval $[1.7188, 1.75]$

Example: Find the root of the following function

$$f(x) = e^{-x}(3.2 \sin(x) - 0.5 \cos(x))$$ on the interval $[3, 4]$.

Solution:

a	b	f(a)	f(b)	c	f(c)
3	4	0.047127	-0.038372	3.5	-0.019757
3	3.5	0.047127	-0.019757	3.25	0.0058479
3.25	3.5	0.0058479	-0.019757	3.375	-0.0086808
3.25	3.375	0.0058479	-0.0086808	3.3125	-0.0018773
3.25	3.3125	0.0058479	-0.0018773	3.2812	0.0018739
3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.00002479
3.2812	3.2968	0.0018739	-0.000024791	3.289	0.00091736
3.289	3.2968	0.00091736	-0.000024791	3.2929	0.00044352

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Example: Find a root of $x^4 - x - 10 = 0$ Using fixed point method?

Solution: Let $g_1(x) = \frac{10}{x^3 - 1}$, with intial guess $x_0 = 2$

i	0	1	2	3	4	5
x_i	2	1.429	5.214	0.071	−10.004	−9.978E-3

- Let $g_2(x) = (x+10)^{1/4}$

$\therefore x_{i+1} = (x_i + 10)^{1/4}$

let the intial guess be 1, 2 and 4

i	0	1	2	3	4
x_i	1	1.82116	1.85424	1.85553	1.85558
x_i	2	1.861	1.8558	1.85559	1.85558
x_i	4	1.93434	1.85866	1.8557	1.85558

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- Let $g3(x) = \dfrac{(x+10)^{1/2}}{x}$, with intial $x_0 = 1.8$

i	0	1	2	3	4	5	....	98
x_i	1.8	1.9084	1.80825	1.90035	1.81529	1.89355		1.8555

Example: Find the root of $(\cos(x)) - (x \exp(x)) = 0$

Solution:

$$\cos(x) = x \exp(x)$$

$$x = \frac{\cos x}{\exp(x)}$$

Let the intial guess $x_0 = 1$

i	0	1	2	3	4	5	----
x_i	1	0.1987	0.803	0.311	0.698	0.381	----

31	32
0.518	0.518

$$\therefore \quad g(x) = \frac{\cos x}{\exp(x)} \quad \text{converges to } 0.518$$

Example: Find the root of $x - \sin(x) - \frac{1}{2} = 0$

Solution: Consider $g(x) = \sin x + 1/2$

Let the intial guess $x_0 = 2$

i	0	1	2	3	4	5
x_i	2	1.409	1.487	1.496	1.497	1.497

$\therefore$ $g(x) = \sin x + \frac{1}{2}$ converges to 1.497

Example: Find the root of $\exp(-x) = 3\log(x)$

Solution: consider $\exp\left[\frac{\exp(-\cancel{x})}{3}\right]$

Let the intial guess be $x_0 = 2$

i	0	1	2	3	4	5	6
x_i	2	1.046	1.124	1.114	1.116	1.115	1.115

$\therefore$ $g(x) = \exp\left[\frac{(\exp(-x))}{3}\right]$ converges to 1.115

System of nonlinear equation - Newton-Raphson method

Example: Solve the following nonlinear equations using Newton-method.

start at $x=1$ and $y=1$

$$4x^2 - y^3 + 28 = 0$$

$$3x^3 + 4y^2 - 145 = 0$$

Solution: The formula of the solution is

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left[J(x_0, y_0) \right]^{-1} \begin{bmatrix} f_1(x_0, y_0) \\ f_2(x_0, y_0) \end{bmatrix}$$

Let $f_1(x,y) = 4x^2 - y^3 + 28 = 0$

$f_2(x,y) = 3x^3 + 4y^2 - 145 = 0$

Now to comput the Jacobian determinent as follows:

$$J(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$f_1 = 4x^2 - y^3 + 28$$

$$\frac{\partial f_1}{\partial x} = 8x \quad \text{and} \quad \frac{\partial f_1}{\partial y} = -3y^2$$

$$f_2 = 3x^3 + 4y^2 - 145$$

$$\frac{\partial f_2}{\partial x} = 9x^2 \quad \text{and} \quad \frac{\partial f_2}{\partial y} = 8y$$

$$J(x,y) = \begin{bmatrix} 8x & -3y^2 \\ 9x^2 & 8y \end{bmatrix}$$

$$\left[J(x,y)\right]^{-1} = \frac{1}{\det J} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{64xy + 27y^2x^2} \begin{bmatrix} 8y & 3y^2 \\ -9x^2 & 8x \end{bmatrix}$$

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$$\left[J(x_0,y_0)\right]^{-1} = \frac{1}{91}\begin{bmatrix} 8 & 3 \\ -9 & 8 \end{bmatrix} = \begin{bmatrix} 0.08791 & 0.0329 \\ -0.0989 & 0.08791 \end{bmatrix}$$

$$f_1(x_0,y_0) = 4(1)^2 - (1)^3 + 28 = 31$$

$$f_2(x_0,y_0) = 3(1)^3 + 4(1)^2 - 145 = -138$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \left[J(x,y)\right]^{-1}\begin{bmatrix} f_1(x_0,y_0) \\ f_2(x_0,y_0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.08791 & 0.03296 \\ -0.0989 & 0.0879 \end{bmatrix}\begin{bmatrix} 31 \\ -138 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2.7252 & -4.549 \\ -3.065 & -12.1315 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1.8238 \\ -15.1975 \end{bmatrix} = \begin{bmatrix} 2.8238 \\ 16.1975 \end{bmatrix}$$

H-W

Solve the following non linear equations using Newton method.

$$\begin{bmatrix} 6.6763 \\ 10.2[illegible]5 \end{bmatrix}$$

$$x^2 - xy + 20 = 0$$

$$y^2 - 2xy + 10 = 0$$

start at $x = 6$ and $y = 10$